Problem 1) a) The average E-field over a spherical shell of radius $r > R$ (i.e., outside the dipole) can readily be shown to vanish, that is,

$$\langle \boldsymbol{E}(\boldsymbol{r})\rangle = \left(\frac{\wp_0}{4\pi\varepsilon_0 r^3}\right)\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}[2\cos\theta\,\overbrace{(\sin\theta\cos\varphi\,\hat{\boldsymbol{x}}+\sin\theta\sin\varphi\,\hat{\boldsymbol{y}}+\cos\theta\,\hat{\boldsymbol{z}})}^{\hat{\boldsymbol{r}}}$$
$$+\sin\theta\,\overbrace{(\cos\theta\cos\varphi\,\hat{\boldsymbol{x}}+\cos\theta\sin\varphi\,\hat{\boldsymbol{y}}-\sin\theta\,\hat{\boldsymbol{z}})}^{\hat{\boldsymbol{\theta}}}]\overbrace{r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi}^{\text{surface element}}$$
$$=\left(\frac{\wp_0\hat{\boldsymbol{z}}}{2\varepsilon_0 r}\right)\int_{\theta=0}^{\pi}(2\sin\theta\cos^2\theta-\underbrace{\sin^3\theta}_{\sin\theta\,(1-\cos^2\theta)})\mathrm{d}\theta=\left(\frac{\wp_0\hat{\boldsymbol{z}}}{2\varepsilon_0 r}\right)\int_0^{\pi}(3\sin\theta\cos^2\theta-\sin\theta)\mathrm{d}\theta$$
$$=\left(\frac{\wp_0\hat{\boldsymbol{z}}}{2\varepsilon_0 r}\right)(\cos\theta-\cos^3\theta)\big|_{\theta=0}^{\pi}=0. \quad (1)$$

Equation (1) reveals that the integral of the dipolar E-field over the entire region outside the sphere of radius R equals zero. In what follows, we will use this result to argue that the E-fields produced by a large number of uniformly distributed spherical dipoles throughout free space average out to zero everywhere except, of course, inside the individual dipoles themselves, where the average E-field produced by all the dipoles equals the local internal dipolar field, namely, $\boldsymbol{E} = -\boldsymbol{P}_0/3\varepsilon_0$.

b) The units of the dipole moment $\boldsymbol{\wp}_0$ are [coulomb · meter], and those of the E-field $E_0\hat{\boldsymbol{z}}$ are [volt/meter]. Thus, the polarizability coefficient ζ has units of [coulomb · meter^2/volt], which is the same as [farad · meter^2].

c) The volume occupied by each spherical particle is $v = 4\pi R^3/3$. Since there are N particles per unit volume, the volume fraction occupied by the (randomly distributed) particles is $4\pi N R^3/3$. As pointed out in part (a), the E-field produced by each particle outside its own sphere averages out to zero, but the internal field $\boldsymbol{E} = -P_0\hat{\boldsymbol{z}}/3\varepsilon_0$ remains. The spatially-averaged E-field over a unit volume of space thus has a contribution $E_0\hat{\boldsymbol{z}}$ from the applied field, and a second contribution, $-(4\pi N R^3/3)\,P_0\hat{\boldsymbol{z}}/3\varepsilon_0 = -N\boldsymbol{\wp}_0/3\varepsilon_0 = -N\zeta E_0\hat{\boldsymbol{z}}/3\varepsilon_0$, from the interiors of all the spherical particles. The spatially averaged E-field over a unit volume is thus given by

$$\langle \boldsymbol{E}(\boldsymbol{r})\rangle = \left(1-\frac{N\zeta}{3\varepsilon_0}\right)E_0\hat{\boldsymbol{z}}. \quad (2)$$

d) The dielectric susceptibility χ_e of the gas is the dimensionless ratio of its average induced polarization, namely, $\langle \boldsymbol{P}(\boldsymbol{r})\rangle = N\zeta E_0\hat{\boldsymbol{z}}$, to ε_0 times the average E-field across the medium, namely, $\varepsilon_0\langle \boldsymbol{E}(\boldsymbol{r})\rangle$; that is,

$$\chi_e = \frac{\langle \boldsymbol{P}(\boldsymbol{r})\rangle}{\varepsilon_0\langle \boldsymbol{E}(\boldsymbol{r})\rangle} = \frac{N\zeta E_0\hat{\boldsymbol{z}}}{\varepsilon_0[1-(N\zeta/3\varepsilon_0)]E_0\hat{\boldsymbol{z}}} = \frac{3(N\zeta/\varepsilon_0)}{3-(N\zeta/\varepsilon_0)}. \quad (3)$$

e) If the "gas" happens to be a homogeneous mixture of K different components, each having number-density N_k and polarizability ζ_k, we will have

$$\chi_e = \frac{\langle \boldsymbol{P}(\boldsymbol{r})\rangle}{\varepsilon_0\langle \boldsymbol{E}(\boldsymbol{r})\rangle} = \frac{3\sum_{k=1}^{K}(N_k\zeta_k/\varepsilon_0)}{3-\sum_{k=1}^{K}(N_k\zeta_k/\varepsilon_0)}. \quad (4)$$

f) The general result obtained in part (e) is similar to the Clausius-Mossotti correction for the susceptibility $\chi_e(\omega)$, with $C_K(\omega)$ of Chapter 6, Eq.(11b), rather than being given by Eq.(8) of

Chapter 6, replaced here by $\sum_{k=1}^{K}(N_k\zeta_k/\varepsilon_0)$. The polarizability ζ_k of type k spheres is, of course, a *static* polarizability associated with a time-independent excitation E-field $E_0\hat{\boldsymbol{z}}$, whereas, in the Lorentz oscillator model, the polarizability $(q_k^2/m_k)/(\omega_{0k}^2-\omega^2-\mathrm{i}\gamma_k\omega)$ is a dynamic property of individual oscillators excited by $E_0\cos(\omega t)\,\hat{\boldsymbol{z}}$, which is an oscillatory field of frequency ω; see Chapter 6, Eq.(4). Despite these differences, the results of the static model remain applicable in many dynamic situations, and the assumptions that led to the above Eqs.(3) and (4) can be shown to remain more or less valid at microwave and even optical frequencies — so long as the particles are sufficiently small, isotropic, and uniformly distributed throughout space.

Problem 2) a) In the incidence medium, both the incident and reflected k-vectors are real, with shared magnitude of k_0n_1 and with $k_x^{(\mathrm{r})}=k_x^{(\mathrm{i})}$. In the transmittance medium, $k^2=k_x^2+k_z^2=k_0^2n_2^2$ and $k_x^{(\mathrm{t})}=k_x^{(\mathrm{i})}$. Therefore,

$$\boldsymbol{k}^{(\mathrm{i})}=k_0n_1\sin\theta\,\hat{\boldsymbol{x}}-k_0n_1\cos\theta\,\hat{\boldsymbol{z}},\tag{1a}$$

$$\boldsymbol{k}^{(\mathrm{r})}=k_0n_1\sin\theta\,\hat{\boldsymbol{x}}+k_0n_1\cos\theta\,\hat{\boldsymbol{z}},\tag{1b}$$

$$\boldsymbol{k}^{(\mathrm{t})}=k_0n_1\sin\theta\,\hat{\boldsymbol{x}}-\sqrt{(k_0n_2)^2-(k_0n_1\sin\theta)^2}\,\hat{\boldsymbol{z}}.\tag{1c}$$

Digression: When $k_z^{(\mathrm{t})}$ is real-valued (i.e., when $n_1\sin\theta\le n_2$), the vector $\boldsymbol{k}^{(\mathrm{t})}$ makes an angle θ' with the normal to the interfacial plane. In this case, $\tan\theta'=|k_x^{(\mathrm{t})}/k_z^{(\mathrm{t})}|=n_1\sin\theta/\sqrt{n_2^2-n_1^2\sin^2\theta}$ and $\cos\theta'=1/\sqrt{1+\tan^2\theta'}=\sqrt{1-(n_1\sin\theta/n_2)^2}$. Consequently, $\sin\theta'=\tan\theta'\cos\theta'=(n_1/n_2)\sin\theta$, and $\boldsymbol{k}^{(\mathrm{t})}=k_0n_2(\sin\theta'\,\hat{\boldsymbol{x}}-\cos\theta'\,\hat{\boldsymbol{z}})$.

b) From Maxwell's 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$, we find, in the absence of free charges, $\boldsymbol{k}\cdot\boldsymbol{D}=\varepsilon_0\varepsilon(\omega)\boldsymbol{k}\cdot\boldsymbol{E}=0$, which yields $\boldsymbol{k}\cdot\boldsymbol{E}=k_xE_{0x}+k_zE_{0z}=0$. Therefore,

$$E_{0z}^{(\mathrm{i})}=(\tan\theta)E_{0x}^{(\mathrm{i})},\tag{2a}$$

$$E_{0z}^{(\mathrm{r})}=-(\tan\theta)E_{0x}^{(\mathrm{r})},\tag{2b}$$

$$E_{0z}^{(\mathrm{t})}=\left(n_1\sin\theta/\sqrt{n_2^2-n_1^2\sin^2\theta}\right)E_{0x}^{(\mathrm{t})}.\tag{2c}$$

c) Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, yields $\boldsymbol{H}_0=\boldsymbol{k}\times\boldsymbol{E}_0/(\mu_0\mu\omega)=(\boldsymbol{k}/k_0)\times\boldsymbol{E}_0/Z_0=(k_zE_{0x}-k_xE_{0z})\hat{\boldsymbol{y}}/(k_0Z_0)$. Consequently,

$$Z_0H_{0y}^{(\mathrm{i})}=-n_1\cos\theta\,E_{0x}^{(\mathrm{i})}-n_1\sin\theta\tan\theta\,E_{0x}^{(\mathrm{i})}=-n_1E_{0x}^{(\mathrm{i})}/\cos\theta,\tag{3a}$$

$$Z_0H_{0y}^{(\mathrm{r})}=n_1\cos\theta\,E_{0x}^{(\mathrm{r})}+n_1\sin\theta\tan\theta\,E_{0x}^{(\mathrm{r})}=n_1E_{0x}^{(\mathrm{r})}/\cos\theta,\tag{3b}$$

$$\begin{aligned}Z_0H_{0y}^{(\mathrm{t})}&=-\sqrt{n_2^2-n_1^2\sin^2\theta}\,E_{0x}^{(\mathrm{t})}-n_1\sin\theta\left(n_1\sin\theta/\sqrt{n_2^2-n_1^2\sin^2\theta}\right)E_{0x}^{(\mathrm{t})}\\&=-n_2^2E_{0x}^{(\mathrm{t})}/\sqrt{n_2^2-n_1^2\sin^2\theta}=-n_2E_{0x}^{(\mathrm{t})}/\sqrt{1-(n_1/n_2)^2\sin^2\theta}.\end{aligned}\tag{3c}$$

d) Continuity of the tangential $\boldsymbol{E}$ and tangential $\boldsymbol{H}$ at the interface between the two media yields

$$E_{0x}^{(\mathrm{i})}+E_{0x}^{(\mathrm{r})}=E_{0x}^{(\mathrm{t})}.\tag{4}$$

$$\begin{aligned}H_{0y}^{(\mathrm{i})}+H_{0y}^{(\mathrm{r})}=H_{0y}^{(\mathrm{t})}\;&\rightarrow\;-(n_1/\cos\theta)E_{0x}^{(\mathrm{i})}+(n_1/\cos\theta)E_{0x}^{(\mathrm{r})}=-n_2E_{0x}^{(\mathrm{t})}/\sqrt{1-(n_1/n_2)^2\sin^2\theta}\\&\rightarrow\;(n_1/n_2)\sqrt{1-(n_1/n_2)^2\sin^2\theta}\left(E_{0x}^{(\mathrm{i})}-E_{0x}^{(\mathrm{r})}\right)=\cos\theta\left(E_{0x}^{(\mathrm{i})}+E_{0x}^{(\mathrm{r})}\right).\end{aligned}\tag{5}$$

These equations can now be solved for the reflection and transmission coefficients, as follows:

$$\rho_p = E_{0x}^{(\mathrm{r})}/E_{0x}^{(\mathrm{i})} = \frac{(n_1/n_2)\sqrt{1-(n_1/n_2)^2\sin^2\theta} - \cos\theta}{(n_1/n_2)\sqrt{1-(n_1/n_2)^2\sin^2\theta} + \cos\theta}, \tag{6}$$

$$\tau_p = E_{0x}^{(\mathrm{t})}/E_{0x}^{(\mathrm{i})} = 1 + \rho_p = \frac{2(n_1/n_2)\sqrt{1-(n_1/n_2)^2\sin^2\theta}}{(n_1/n_2)\sqrt{1-(n_1/n_2)^2\sin^2\theta} + \cos\theta}. \tag{7}$$

e) At the Brewster angle θ_B, we have $\rho_p = 0$. Therefore,

$$(n_1/n_2)\sqrt{1-(n_1/n_2)^2\sin^2\theta_B} - \cos\theta_B = 0 \;\rightarrow\; (n_1/n_2)^2[1-(n_1/n_2)^2\sin^2\theta_B] = \cos^2\theta_B$$

$$\rightarrow\; (1+\tan^2\theta_B) - (n_1/n_2)^2\tan^2\theta_B = (n_2/n_1)^2$$

$$\rightarrow\; \tan^2\theta_B = \frac{(n_2/n_1)^2 - 1}{1-(n_1/n_2)^2} = (n_2/n_1)^2 \quad\rightarrow\quad \theta_B = \tan^{-1}(n_2/n_1). \tag{8}$$

f) Total internal reflection (TIR) occurs when the expression under the square root in Fresnel's formulas becomes negative, that is, when $(n_1/n_2)\sin\theta > 1$. Under this circumstance, we will have $k_z^{(\mathrm{t})} = -\mathrm{i}k_0 n_2\sqrt{(n_1\sin\theta/n_2)^2 - 1}$, $E_{0z}^{(\mathrm{t})} = -\mathrm{i}\,(n_1\sin\theta/n_2)E_{0x}^{(\mathrm{t})}/\sqrt{(n_1\sin\theta/n_2)^2-1}$, and $Z_0H_{0y}^{(\mathrm{t})} = \mathrm{i}n_2E_{0x}^{(\mathrm{t})}/\sqrt{(n_1\sin\theta/n_2)^2-1}$, and the Fresnel coefficients of Eqs.(6), (7) become

$$\rho_p = E_{0x}^{(\mathrm{r})}/E_{0x}^{(\mathrm{i})} = \frac{\mathrm{i}(n_1/n_2)\sqrt{(n_1\sin\theta/n_2)^2-1} - \cos\theta}{\mathrm{i}(n_1/n_2)\sqrt{(n_1\sin\theta/n_2)^2-1} + \cos\theta}, \tag{9}$$

$$\tau_p = E_{0x}^{(\mathrm{t})}/E_{0x}^{(\mathrm{i})} = \frac{2\mathrm{i}(n_1/n_2)\sqrt{(n_1\sin\theta/n_2)^2-1}}{\mathrm{i}(n_1/n_2)\sqrt{(n_1\sin\theta/n_2)^2-1} + \cos\theta}. \tag{10}$$

g) The Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$ is the cross-product of the real-valued $\boldsymbol{E}(\boldsymbol{r},t)$ and the real-valued $\boldsymbol{H}(\boldsymbol{r},t)$. In addition to the complex field amplitudes $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, the complex exponential factor $\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ has real and imaginary parts, which need to be taken into account. We write

$$\boldsymbol{S}^{(\mathrm{t})}(\boldsymbol{r},t) = \boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t)\times\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = \mathrm{Re}\left\{(E_{0x}^{(\mathrm{t})}\hat{\boldsymbol{x}} + E_{0z}^{(\mathrm{t})}\hat{\boldsymbol{z}})\exp[\mathrm{i}(k_x^{(\mathrm{t})}x + k_z^{(\mathrm{t})}z - \omega t)]\right\}$$
$$\times\,\mathrm{Re}\left\{H_{0y}^{(\mathrm{t})}\hat{\boldsymbol{y}}\exp[\mathrm{i}(k_x^{(\mathrm{t})}x + k_z^{(\mathrm{t})}z - \omega t)]\right\}. \tag{11}$$

In what follows, we consider two possible situations corresponding to $\theta \le \theta_c$ and $\theta > \theta_c$, where θ_c is the critical angle of total internal reflection.

Case I: $\theta \le \theta_c$. Here, we have $E_{0x}^{(\mathrm{t})} = |E_{0x}^{(\mathrm{t})}|e^{\mathrm{i}\varphi_{0x}}$, $k_x^{(\mathrm{t})} = k_x^{(\mathrm{i})} = k_0 n_1\sin\theta$ and, recalling Eqs.(1c), (2c), and (3c),

$$k_z^{(\mathrm{t})} = -k_0 n_2\sqrt{1-(n_1\sin\theta/n_2)^2}, \tag{12a}$$

$$E_{0z}^{(\mathrm{t})} = \frac{n_1\sin\theta/n_2}{\sqrt{1-(n_1\sin\theta/n_2)^2}}|E_{0x}^{(\mathrm{t})}|e^{\mathrm{i}\varphi_{0x}}, \tag{12b}$$

$$H_{0y}^{(\mathrm{t})} = -\frac{n_2}{Z_0\sqrt{1-(n_1\sin\theta/n_2)^2}}|E_{0x}^{(\mathrm{t})}|e^{\mathrm{i}\varphi_{0x}}. \tag{12c}$$

Consequently,

$$\boldsymbol{S}^{(\mathrm{t})}(\boldsymbol{r},t) = |E_{0x}^{(\mathrm{t})}|\left[\hat{\boldsymbol{x}} + \frac{n_1\sin\theta/n_2}{\sqrt{1-(n_1\sin\theta/n_2)^2}}\hat{\boldsymbol{z}}\right]\times\left[-\frac{n_2|E_{0x}^{(\mathrm{t})}|}{Z_0\sqrt{1-(n_1\sin\theta/n_2)^2}}\hat{\boldsymbol{y}}\right]$$
$$\times\cos^2(k_x^{(\mathrm{t})}x + k_z^{(\mathrm{t})}z - \omega t + \varphi_{0x})$$

$$= Z_0^{-1}\left|E_{0x}^{(\mathrm{t})}\right|^2\left[\frac{n_1\sin\theta}{1-(n_1\sin\theta/n_2)^2}\hat{\boldsymbol{x}}-\frac{n_2}{\sqrt{1-(n_1\sin\theta/n_2)^2}}\hat{\boldsymbol{z}}\right]$$

$$\times\cos^2[(k_0 n_1\sin\theta)x-k_0\sqrt{n_2^2-n_1^2\sin^2\theta}\,z-\omega t+\varphi_{0x}]. \qquad (13)$$

Case II: $\theta > \theta_c$. Again, we write $E_{0x}^{(\mathrm{t})} = |E_{0x}^{(\mathrm{t})}|e^{\mathrm{i}\varphi_{0x}}$, $k_x^{(\mathrm{t})} = k_x^{(\mathrm{i})} = k_0 n_1\sin\theta$ and, recalling Eqs.(1c), (2c), and (3c),

$$k_z^{(\mathrm{t})} = -\mathrm{i}k_0 n_2\sqrt{(n_1\sin\theta/n_2)^2-1}, \qquad (14a)$$

$$E_{0z}^{(\mathrm{t})} = -\frac{\mathrm{i}(n_1\sin\theta/n_2)}{\sqrt{(n_1\sin\theta/n_2)^2-1}}|E_{0x}^{(\mathrm{t})}|e^{\mathrm{i}\varphi_{0x}}, \qquad (14b)$$

$$H_{0y}^{(\mathrm{t})} = \frac{\mathrm{i}n_2}{Z_0\sqrt{(n_1\sin\theta/n_2)^2-1}}|E_{0x}^{(\mathrm{t})}|e^{\mathrm{i}\varphi_{0x}}. \qquad (14c)$$

Consequently,

$$\boldsymbol{S}^{(\mathrm{t})}(\boldsymbol{r},t) = \exp(\mathrm{i}k_z^{(\mathrm{t})}z)|E_{0x}^{(\mathrm{t})}|\,\mathrm{Re}\left\{\left[\hat{\boldsymbol{x}}-\frac{\mathrm{i}(n_1\sin\theta/n_2)}{\sqrt{(n_1\sin\theta/n_2)^2-1}}\hat{\boldsymbol{z}}\right]\exp[\mathrm{i}(k_x^{(\mathrm{t})}x-\omega t+\varphi_{0x})]\right\}$$

$$\times\exp(\mathrm{i}k_z^{(\mathrm{t})}z)|E_{0x}^{(\mathrm{t})}|\,\mathrm{Re}\left\{\frac{\mathrm{i}n_2}{Z_0\sqrt{(n_1\sin\theta/n_2)^2-1}}\hat{\boldsymbol{y}}\exp[\mathrm{i}(k_x^{(\mathrm{t})}x-\omega t+\varphi_{0x})]\right\}$$

$$= \exp(\mathrm{i}2k_z^{(t)}z)\,|E_{0x}^{(\mathrm{t})}|^2\left[\cos(k_x^{(\mathrm{t})}x-\omega t+\varphi_{0x})\,\hat{\boldsymbol{x}}+\frac{(n_1\sin\theta/n_2)\sin(k_x^{(\mathrm{t})}x-\omega t+\varphi_{0x})}{\sqrt{(n_1\sin\theta/n_2)^2-1}}\hat{\boldsymbol{z}}\right]$$

$$\times\left[-\frac{n_2\sin(k_x^{(\mathrm{t})}x-\omega t+\varphi_{0x})}{Z_0\sqrt{(n_1\sin\theta/n_2)^2-1}}\hat{\boldsymbol{y}}\right]$$

$$= Z_0^{-1}\left|E_{0x}^{(\mathrm{t})}\right|^2\exp(2k_0\sqrt{n_1^2\sin^2\theta-n_2^2}\,z)$$

$$\times\left[\frac{n_1\sin\theta\sin^2(k_0 n_1\sin\theta x-\omega t+\varphi_{0x})}{(n_1\sin\theta/n_2)^2-1}\hat{\boldsymbol{x}}-\frac{n_2\sin[2(k_0 n_1\sin\theta x-\omega t+\varphi_{0x})]}{2\sqrt{(n_1\sin\theta/n_2)^2-1}}\hat{\boldsymbol{z}}\right]. \qquad (15)$$

h) The time-averaged Poynting vector in Case II above is readily found to be

$$\langle\boldsymbol{S}^{(\mathrm{t})}(\boldsymbol{r},t)\rangle = \frac{n_1\sin\theta}{2Z_0[(n_1\sin\theta/n_2)^2-1]}\exp(2k_0\sqrt{n_1^2\sin^2\theta-n_2^2}\,z)\left|E_{0x}^{(\mathrm{t})}\right|^2\hat{\boldsymbol{x}}. \qquad (16)$$

The time-averaged Poynting vector is seen to have only an x-component. Moreover, this component decays exponentially with the distance z from the interface between media 1 and 2.